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On the Asymptotic Structure of the Polynomials of Minimal Diophantic Deviation from Zero

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It is proved that, for all $N > N_1$, every polynomial with minimal (uniform) diophantic deviation from zero in $[0, 1]$ is as follows,

$$P_N(x) = [x(1-x)]^{[N\lambda_1]} (2x-1)^{[N\lambda_2]} (5x^2-5x+1)^{[N\lambda_3]} Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients and $0.1456 < \lambda_1 < 0.1495$, $0.0166 < \lambda_2 < 0.0187$, $0.0037 < \lambda_3 < 0.0053$. Also, two general theorems for the case of the arbitrary intervals are demonstrated.

1. INTRODUCTION

Let H_n be the set of all polynomials of degree $m \leqslant n$ with integral coefficients not simultaneously zero:

$$P(x) = \sum_{k=0}^{m} c_k x^k. \tag{1}$$

We write

$$\rho_n^{-n} = \min_{P \in H_n} \max_{a \leqslant x \leqslant b} |P(x)|, \tag{2}$$

$$\rho = \lim_{n \to \infty} \rho_n. \tag{3}$$

The existence of the limit (3) was proved by L. G. Shnirelman (see [2, 3]).

The polynomials $P(x) \in H_n$ for which the minimum (2) is attained, that is, such that

$$\max_{a \leqslant x \leqslant b} |P(x)| = \rho_n^{-n}, \tag{4}$$

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are named polynomials of minimal diophantic (uniform) deviation from zero in the interval $[a, b]$.

The problem of the asymptotic structure of those polynomials has been proposed by A. O. Gelfond (see [2, 4, 6]).

2. Fundamental Theorems

Theorem 1. *Let*

$$f(x)=a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{5}$$

be a primitive polynomial with integer coefficients such that $(a_0, a_1, ..., a_n)=1$. *Suppose that its zeros* $x_1, x_2, ..., x_n$ *are real and belong to the interval* $[a, b]$, $a \leqslant x_i \leqslant b$, $i=1, 2, ..., n$. *Let*

$$a_n = p_1^{\alpha_{1n}} \cdots p_v^{\alpha_{vn}}, \qquad \alpha_{kn} \geqslant 1, k=1, ..., v, \tag{6}$$

be the canonical decomposition in prime factors of the coefficient a_n *and suppose that*

$$a_m = p_1^{\alpha_{1m}} \cdots p_v^{\alpha_{vm}} b_m, \tag{7}$$

where $\alpha_{km} \geqslant 0$, $(b_m, a_n)=1$, $m=0, 1, ..., n-1$, $k=1, 2, ..., v$.

Consider the rational numbers

$$\lambda_k = \max_{m=0, 1, ..., n-1} \frac{n}{n-m} (\alpha_{kn} - \alpha_{km}) - \alpha_{kn}, \tag{8}$$

$$t_k = p_k^{(\lambda_k + \alpha_{kn})/n}, \qquad k=1, 2, ..., v. \tag{9}$$

If the inequality

$$T = \prod_{k=1}^{v} t_k < \rho, \tag{10}$$

is verified, then for $N > N_1$ *large enough, every polynomial* $P_N(x)$ *of minimal diophantic (uniform) deviation from zero in the interval* $[a, b]$ *vanishes on the roots of the polynomial* $f(x)$.

Proof. Multiplying $f(x)$ by

$$A = p_1^{\lambda_1} \cdots p_v^{\lambda_v}, \tag{11}$$

we have

$$Af(x) = z^n + \sum_{m=0}^{n-1} \gamma_m z^m, \tag{12}$$

where $z = Tx$ and

$$\gamma_m = b_m \prod_{k=1}^{v} p_k^{(n-m)((\lambda_k + \alpha_{kn})/n - (\alpha_{kn} - \alpha_{km})/(n-m))},$$

From the choice of the numbers λ_k, the exponents of p_k in (9) are rational non-negative numbers, and therefore each γ_m is an algebraic integer. Then, the zeros of the polynomial (12)

$$z_1 = Tx_1, ..., z_n = Tx_n \tag{13}$$

are algebraic numbers as well.

Let

$$P_N(x) = A_N x_N + A_{N-1} x^{N-1} + \cdots + A_1 x + A_0 \tag{14}$$

be a polynomial of degree N with rational integer coefficients and minimal deviation from zero in $[a, b]$, so that

$$\rho_N^{-N} = \max_{a \leqslant x \leqslant b} |P_N(x)|. \tag{15}$$

Consider the expression

$$B_\mu = \left| \sum_{k=1}^{n} x_k P_N(x_k) \right| \prod_{k=1}^{v} t_k^{N+\mu+m_{k\mu}}, \qquad \mu = 0, 1, ..., n-1, \tag{16}$$

where the integers $0 \leqslant m_{k\mu} \leqslant n$, $k = 1, 2, ..., v$, have been chosen such that the numbers

$$\frac{\lambda_k + \alpha_{kn}}{n} (N + \mu + m_{k\mu})$$

are integers. Then expression (16) can be written as

$$\left| \sum_{k=1}^{n} (Tx_k)^\mu \left[A_N (Tx_k)^N + A_{N-1} T(Tx_k)^{N-1} + \cdots + A_0 T^N\right] \right| \prod_{k=1}^{v} t_k^{m_{k\mu}},$$

$\mu = 0, 1, ... n-1$. Every number in this formula is an algebraic integer, and so they are algebraic integers as well.

By virtue of an appropiate choice of $m_{k\mu}$, $k = 1, 2, ..., v$, the numbers

$$t_k^{N+\mu+m_k} = p_k^{((\alpha_{kn} + \lambda_k)/n)(N+\mu+m_{k\mu})}, \qquad k = 1, 2, ..., v,$$

are rational integers, and consequently, the factor of (16),

$$\prod_{k=1}^{v} t_k^{N+\mu+m_{k\mu}},$$

is a rational integer.

But the sum

$$\sum_{k=1}^{n} x_k^{\mu} P_N(x_k),$$

is a rational number since it is a symmetric entire function of the zeros of the polynomial $f(x)$.

Hence B_μ is a rational number for each $\mu = 0, 1, ..., n-1$ and it is an algebraic integer as well; therefore, it is a nonnegative rational integer, $B_\mu \geqslant 0$.

Now, we shall obtain an upper asymptotic bound of B_μ.

It is clear that for $\varepsilon > 0$ small enough, there exists an integer N_0 such that

$$\rho_N > \rho - \varepsilon, \qquad \forall N > N_0.$$

As the roots of the polynomial (5) are in $[a, b]$, we have

$$|P_N(x_k)| \leqslant \rho_N^{-N} < \frac{1}{(\rho-\varepsilon)^N}, \qquad \forall N > N_0, k = 1, 2, ..., n.$$

Hence, for the numbers in (16) we have

$$B_\mu \leqslant K^\mu n \left(\prod_{k=1}^{v} t_k^{\mu+m_{k\mu}} \right) \left(\frac{T}{\rho-\varepsilon} \right)^N < 1, \qquad \forall N > N_1,$$

where N_1 is large enough, $K = \max(|a|, |b|)$, and $\varepsilon > 0$ is chosen such that

$$T < \rho - \varepsilon.$$

As $B_\mu \geqslant 0$ is a rational integer, we derive that $B_\mu = 0$ and so from (16) follows

$$\sum_{k=1}^{n} x_k P_N(x_k) = 0, \qquad \mu = 0, 1, ..., n-1, \tag{17}$$

for every $N > N_1$.

Suppose that

$$x_1, ..., x_q, \qquad q \leqslant n,$$

are the roots of the polynomial $f(x)$ and $n_1, ..., n_q$ are their multiplicities, respectively, so that $n_1 + \cdots + n_q = n$. We can write (17) as

$$\sum_{k=1}^{q} n_k x_k^\mu P_N(x_k) = 0, \qquad \mu = 0, 1, ..., n-1,$$

as q is the rank of the matrix $\| n_k x_k^\mu \|$, and it follows that

$$P_N(x_k) = 0, \qquad k = 0, 1, ..., q.$$

The theorem is proved.

We now present some particular cases.

If $a_n = 1$, condition (10) is

$$\rho > 1. \tag{18}$$

If $a_n = p$ is prime, then condition (10) takes the form

$$T = p^{1/(n-m)} < \rho, \tag{19}$$

where m is the maximum degree of the terms in (5) with coefficients satisfying the condition $(a_m, p) = 1$.

3. Auxiliar-Lemma

In 1892, V. A. Markov [10] proved that for the derivative of order r of a polynomial $P_n(x)$ of degree n with real coefficients the following inequality is satisfied,

$$\max_{a \leqslant x \leqslant b} |P_n^{(r)}(x)| \leqslant M \frac{2^r}{(b-a)^r} \cdot \frac{n^2(n^2-1^2)(n^2-2^2) \cdots (n^2-(r-1)^2)}{(2r-1)!!}, \tag{20}$$

where $M = \max_{a \leqslant x \leqslant b} |P_n(x)|$.

By means of this inequality, we shall show the following lemma which will be used later.

Lemma. *The derivative of order $r = \lambda n$, $0 < \lambda < 1$, of every polynomial $P_n(x)$ of degree n and real coefficients satisfies the inequality*

$$\max_{a \leqslant x \leqslant b} |P_n^{(r)}(x)| < M \frac{r!}{(b-a)^r} \cdot \frac{C(\lambda)}{\sqrt{n}} L^n(\lambda), \tag{21}$$

where

$$C(\lambda)=\frac{e^{1/12}}{2\sqrt{\pi\lambda(1-\lambda^2)}}, \qquad L(\lambda)=\frac{(1+\lambda)^{1+\lambda}}{(1-\lambda)^{1-\lambda}\lambda^{2\lambda}}, \tag{22}$$

and $M=\max_{a\leqslant x\leqslant b}|P_n(x)|$.

Proof. It is easy to verify that V. A. Markov's inequality (20) can be written in the form

$$\max_{a\leqslant x\leqslant b}\frac{|P_n^{(r)}(x)|}{r!}<M\frac{4^r}{(b-a)^r}\cdot\frac{n}{n+r}\binom{n+r}{n-r}. \tag{23}$$

Applying the Stirling formula, we obtain

$$\binom{n+r}{n-r}<\frac{(n+r)^{n+r}}{(n-r)^{n-r}(2r)^{2r}}\sqrt{\frac{n+r}{4\pi r(n-r)}}\,e^{1/12}.$$

By means of the substitution $r=\lambda n$, we write

$$\binom{n+r}{n-r}<\frac{e^{1/12}(1+\lambda)}{2\sqrt{\pi\lambda(1-\lambda^2)}}\cdot\frac{1}{4^{\lambda n}\sqrt{n}}L^n(\lambda).$$

and so, from (23), we obtain the inequality (21).

4. Bounds for the Multiplicity Orders

Now we shall apply (21) to find bounds for the multiplicity orders of the roots $x_1, ..., x_q$ of the polynomials $P_N(x)$.

THEOREM 2. *Suppose the conditions of Theorem* 1 *are satisfied. Let r be an integer such that* $1\leqslant r<N$, $r=N\lambda$, $0<\lambda<1$.

If

$$\frac{L(\lambda)}{(b-a)^\lambda}<\frac{\rho}{T^{1-\lambda}}, \tag{24}$$

then each root of the polynomial $f(x)$ *is a zero of the polynomials* $P_N(x)$ *of the above theorem with multiplicity order greater than or equal to* $r=N\lambda$.

Proof. As $(1/r!)\,P_N^{(r)}(x)$ has integer coefficients, by applying the above argument to the expression

$$\frac{1}{r!}\left|\sum_{k=1}^{n}x_kP_N^{(r)}(x_k)\right|\prod_{k=1}^{v}t_k^{N-r+\mu+m_{k\mu}}, \qquad \mu=0, 1, ..., n-1, \tag{25}$$

we derive that they are rational integers.

By virtue of inequality (21), the numbers in (25) are dominated by

$$K^{\mu} n \left(\prod_{k=1}^{v} t_k^{\mu + m_{k\mu}} \right) \frac{C(\lambda)}{\sqrt{N}} \left| \frac{T^{1-\lambda} L(\lambda)}{\rho_N (b-a)^{\lambda}} \right|^N,$$

and this is less than

$$K^{\mu} n \left(\prod_{k=1}^{v} t_k^{\mu + m_{k\mu}} \right) \frac{C(\lambda)}{\sqrt{n}} \left| \frac{T^{1-\lambda} L(\lambda)}{(\rho - \varepsilon)(b-a)^{\lambda}} \right|^N < 1, \qquad \forall N > N_1,$$

where N_1 is large enough and $\varepsilon > 0$ is chosen such that the following inequality holds:

$$\frac{T^{1-\lambda} L(\lambda)}{(b-a)^{\lambda}} < \rho - \varepsilon.$$

Hence, as before, we deduce

$$\sum_{k=1}^{n} x_k^{\mu} P_N^{(r)}(x_k) = 0, \qquad \mu = 0, 1, ..., n-1,$$

for every $N > N_1$, and so

$$P_N^{(r)}(x_k) = 0, \qquad k = 1, 2, ..., q.$$

The theorem is proved.

5. Case of the Interval [0, 1]

In the case of the interval [0, 1], it is known [5] that $2.33071 < \rho < 2.37686$. Obviously, the polynomials

$$x, x-1, 2x-1, 5x^2 - 5x + 1, \tag{26}$$

satisfy the conditions of Theorem 1 since their roots are in [0, 1] and the corresponding values of T are 1, 1, 2, $\sqrt{5}$, respectively, all of them less than ρ. Consequently, the polynomials (26) are divisors of every polynomial of minimal diophantic (uniform) deviation from zero in the interval [0, 1] for $N > N_1$, with N_1 large enough. Thus, we can write

$$P_N(x) = x^{r_1} (1-x)^{r_2} (2x-1)^{r_3} (5x^2 - 5x + 1)^{r_4} Q(x), \qquad \forall N > N_1, \tag{27}$$

where $r_i \geqslant 1$, $i = 1, 2, 3, 4$, and $Q(x)$ is a polynomial with integer coefficients. We now shall calculate the multiplicity orders r_i of the polynomials (26) in the decomposition (27).

The inequalities corresponding to (24) for these polynomials are

$$L(\lambda) < \rho,\ L(\lambda) < \rho,\ L(\lambda) < \frac{\rho}{2^{1-\lambda}},\ L(\lambda) < \frac{\rho}{5^{(1-\lambda)/2}}. \tag{28}$$

Let λ_1^*, λ_2^*, λ_3^* be the respective roots of the equations

$$L(\lambda) = \rho, \qquad L(\lambda) = \frac{\rho}{2^{1-\lambda}}, \qquad L(\lambda) = \frac{\rho}{5^{(1-\lambda)/2}}, \tag{29}$$

which are in the interval (0, 1). Applying Theorem 2, we derive the following statement:

THEOREM 3. *For any rational numbers $\lambda_1, \lambda_2, \lambda_3$, such that $0 < \lambda_1 < \lambda_1^*$, $0 < \lambda_2 < \lambda_2^*$, $0 < \lambda_3 < \lambda_3^*$, and every $N > N_1$, each polynomial with integer coefficients $P_N(x)$ of minimal (uniform) deviation from zero in the interval [0, 1] has the form*

$$P_N(x) = [x(1-x)]^{[N\lambda_1]} (2x-1)^{[N\lambda_2]} (5x^2-5x+1)^{[N\lambda_3]} Q(x), \tag{30}$$

where $Q(x)$ is a polynomial with integer coefficients. Here $[N\lambda_i]$ denotes the integer part of $N\lambda_i$ and N_1 is large enough.

In [11] Sanov observes that it is possible to show that $\rho > 2.343$. By using this lower bound, we obtain

$$\lambda_1^* = 0.1456..., \qquad \lambda_2^* = 0.0166..., \qquad \lambda_3^* = 0.0037... \tag{31}$$

On the other hand, by means of the upper bound $\rho < 2.37686$, we obtain

$$\lambda_1'^* = 0.1494..., \qquad \lambda_2'^* = 0.0186..., \qquad \lambda_3'^* = 0.0052..., \tag{32}$$

which means that the numbers λ_1, λ_2, λ_3 in (30) can be respectively considered such that $\lambda_1^* < \lambda_i < \lambda_i^*$, $i = 1, 2, 3$.

Finally, we recommend that the reader consult [7], which contains an extensive bibliography of the topic of this paper (see also [1, 8, 9, 12]).

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